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Formation of Special Diophantine Quadruples with Property $D(6kpq)^2$

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Abstract:
This paper concerns with the study of constructing a special non-zero integer quadruple (a,b,c,d) such that the product of any two elements of the set increased by a square is a perfect square. Different relations between the elements of the quadruple and special numbers are presented.

***Key words**: Diophantine Quadruple, System of Equations*
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1. Introduction

The problem of constructing the sets with property that product of any two of its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus [1]. For an extensive review of various articles one may refer [2-17]. In this paper, starting with the Diophantine pair (a,b) with the property $D(k^2s^2)$, we extend it to Diophantine triple with property $D(k^2s^2)$ and quadruple with property $D(6kpq)^2$.

2. Notations

- $t_{m,n}$: Polygonal Number of rank n with size m
- SO_n : Stella Octangular Number of rank n
- $\Pr_n$: Pronic Number of rank n
- OH_n : Octahedral Number of rank n
- Pt_n : Pentatope Number of rank n
- P_p^3 : Triangular Pyramidal Number of rank p
- P_p^4 : Square Pyramidal Number of rank p
- P_p^5 : Pentagonal Pyramidal Number of rank p

3. Method of Analysis

Let $a = r - ks, b = r + ks$, where r and s are non-zero distinct integers and the product ab is square free, be any two non-zero integers and $r \neq ks$. Observe that (a,b) is a Diophantine double with property $D(k^2s^2)$

Let c be any non-zero integer such that

$$ac + k^2 s^2 = \alpha^2 \tag{1}$$

$$bc + k^2 s^2 = \beta^2 \tag{2}$$

where $\alpha = 2r - ks$ and $\beta = 2r + ks$

Eliminating c between (1) and (2) we get

$$b\alpha^2 - a\beta^2 = k^2 s^2 (b - a)$$

The choice $\alpha = X + aT, \beta = X + bT$ (3)

leads the above equation to the Pell equation

$$X^2 = abT^2 + k^2 s^2 \tag{4}$$

whose initial solution is

$$T_0 = 1, X_0 = r \tag{5}$$

Using (5) in (3) and employing either (1) or (2) we get

$$c = 4r$$

Therefore (a, b, c) is a Diophantine triple with the property $D(k^2 s^2)$

The triple can be extended to a quadruple as follows.

Let d be any non-zero integer such that

$$ad + k^2 s^2 = \overline{\alpha}^2 \tag{6}$$

$$bd + k^2 s^2 = \overline{\beta}^2 \tag{7}$$

$$cd + k^2 s^2 = \overline{\gamma}^2 \tag{8}$$

Eliminating d between (7) and (8) w get

$$c\overline{\beta}^2 - b\overline{\gamma}^2 = k^2 s^2 (c - b) \tag{9}$$

Taking the linear transformations

$$\overline{\beta} = X + bT, \overline{\gamma} = X + cT \tag{10}$$

in (9) it becomes

$$X^2 = bcT^2 + k^2 s^2 \tag{11}$$

whose initial solution is

$$T_0 = 1, X_0 = \beta$$

Substituting the above values in (10) and employing (7) we get

$$\overline{\beta} = \beta + b$$

which gives $bd = (3r + 3ks)(3r + ks)$

and hence

$$d = 9r + 3ks \tag{12}$$

Using (12) in (6) and simplifying we have

$$(3r - ks)^2 = 3k^2 s^2 + \overline{\alpha}^2 \tag{13}$$

which is satisfied by

$$ks = 2pq, \overline{\alpha} = 3p^2 - q^2, 3r - ks = 3p^2 + q^2 \tag{14}$$

Since our thrust is on integers, note that r and s are integers when q is replaced by $3qk$

Thus,

$$r = p^2 + 3k^2 q^2 + 2kpq$$

$$s = 6pq$$

$$d = 9r + 3ks = 9p^2 + 27k^2 q^2 + 36kpq$$

Therefore (a, b, c, d) is a Diophantine quadruple with the property $D(6kpq)^2$, where

$$a = p^2 + 3k^2 q^2 - 4kpq$$

$$b = p^2 + 3k^2 q^2 + 8kpq$$

$$c = 4p^2 + 12k^2 q^2 + 8kpq$$

Some numerical examples are presented below.

k	p	q	(a,b,c,d)	Property $D(6kpq)^2$
3	2	1	$(7,79,172,495)$	$D(36)^2$
1	4	5	$(11,251,524,1539)$	$D(120)^2$
1	6	7	$(15,519,1068,3159)$	$D(252)^2$

Denoting a,b,c,d by $a(k,p,q), b(k,p,q), c(k,p,q), d(k,p,q)$ respectively, the following relations are observed.

1. $d(k,p,q) = 6b(k,p,q) + 3a(k,p,q)$
2. $c(k,p,q) = 2a(k,p,q) + 2b(k,p,q)$
3. $d(k,p,q) = a(k,p,q) + 4b(k,p,q) + c(k,p,q)$
4. $6c(k,p,q) = 9a(k,p,q) + 6b(k,p,q) + d(k,p,q)$
5. $4b(k,p,p+1) - c(k,p,p+1) = 48kt_{3,p}$
6. $4b(k,p(p+1),p+2) - c(k,p(p+1),p+2) = 144kP_p^3$
7. $4b(k,p(p+1),2p+1) - c(k,p(p+1),2p+1) = 144kP_p^4$
8. $b(k,p,kp(p+1)) - a(k,p,kp(p+1)) = 24k^2P_p^5$
9. Each of the following expressions is a Nasty Number
   - $c(k,kp,p)$
   - $2b(k,kp,p)$
   - $d(k,3kq,q)$
   - $4b(k,p,kp) - c(k,p,kp)$
   - $b(k,2kq,q) - a(k,2kq,q)$

Note: One may also write the solution of (13) as

$$s = 2pq$$

$$r = \frac{3k^2p^2 + q^2 + 2kpq}{3}$$

$$\overline{\alpha} = 3(k^2p^2) - q^2$$

For this choice, the corresponding quadruple with property $D(6kpq)^2$ is $((kp)^2 + 3q^2 - 4kpq, (kp)^2 + 3q^2 + 8kpq, 4(kp)^2 + 12q^2 + 8kpq, 9(kp)^2 + 27q^2 + 72pq)$ Some numerical examples of the above with $k = 2$ are presented below.

p	q	(a,b,c,d)	Property $D(6kpq)^2$
1	3	$(32,272,608,1728)$	$D(36)^2$
4	9	$(19,883,1804,5355)$	$D(432)^2$
5	11	$(23,1343,2732,8127)$	$D(660)^2$
6	13	$(27,1899,3852,11475)$	$D(936)^2$
7	15	$(31,2551,5164,15399)$	$D(1260)^2$

Denoting a,b,c,d by $a(p,q), b(p,q), c(p,q), d(p,q)$ respectively, the following relations are observed.

1. $b(n^2,n+1) - a(n^2,n+1) - 48\,\Pr_n^5 = 0$
2. $c(n,1) - a(n^2,n+1) - t_{20,n} - 16\,\Pr_n^5 - 18p - 9 = 0$
3. $d(n,n) - 9a(n,1) - t_{200,n} = -27 \pmod{106}$

4. $c(n,n+1)-b(n,2n^2-1)-t_{146,n}-16\,\mathrm{Pr}_n^5+160\,SO_n+12t_{4,n^2}\equiv 9 \pmod{44}$
5. $d(n,2n^2+1)-c(n,n+1)-t_{234,n}+16\,\mathrm{Pr}_n-216\,OH_n-108t_{4,n^2}\equiv 15 \pmod{91}$
6. $b(n(n+1),(n+2)(n+3))-a(n(n+1),(n+2)(n+3))-576\,Pt_n=0$
7. $c(1,n)-t_{26,n}-5p-16=0$
8. Each of the following expressions is a Nasty Number
   - $2c(2n,n)$
   - $b(4n,4n)-c(2p,2p)$
   - $b(n,n)-a(n,n)$

4. Conclusion

In the construction of the quadruple we have assumed the product ab is square free. One may assume that the product ab is a perfect square and search for Diophantine quadruples with suitable property.

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