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Relations between M-Gonal Numbers through the Solution of The Equation $Z^2 = 8X^2 + Y^2$

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Abstract:
The ternary quadratic equation given by $Z^2 = 8X^2 + Y^2$ is considered. Employing its non-zero integral solutions, relations among few special polygonal numbers are determined.

Key words*: Pell Equations, Ternary Quadratic Equation*
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1. Introduction

In [1-3], different patterns of m-gonal numbers are presented. In [4] explicit formulas for the rank of Triangular numbers which are simultaneously equal to Pentagonal, Decagonal and Dodecagonal numbers in turn are presented. In [5] the relations among the pairs of special m-gonal numbers generated through the solutions of the ternary quadratic equation $X^2 = 8\alpha^2 + Y^2$ are determined. In [6] the relations among few polygonal and centered polygonal numbers generated through the solutions of $y^2 = 2x^2 + z^2$ are determined.

In this communication, we consider the ternary quadratic equation given by $Z^2 = 8X^2 + Y^2$ and obtain the relations among the pairs of special m-gonal numbers different from [5] generated through the solutions of the equation under consideration.

2. Notations

- $T_{m,n}$ **:** Polygonal number of rank n with m sides
- $Ct_{m,n}$ **:** Centered Polygonal number of rank n with m sides

3. Method of Analysis

Consider the Diophantine equation

$$Z^2 = 8X^2 + Y^2 \quad (1)$$

whose general solutions are

$$\left.\begin{array}{l} X = 8rs \\ Y = 8r^2 - s^2 \\ Z = 8r^2 + s^2 \end{array}\right\} \quad (2)$$

where r and s are non-zero positive integers.

Case (1):
The choice,

$$\boxed{2M + 1 = 8r^2 + s^2}, \boxed{4N - 1 = 8r^2 - s^2} \quad (3)$$

in (1) leads to the relation that

“$T_{3,M} - T_{6,N} =$ **a square integer”**

From (3), the values of ranks of the Triangular numbers and Hexagonal numbers are respectively given by

$$M = \frac{8r^2 + s^2 - 1}{2}, \quad N = \frac{8r^2 - s^2 - 1}{4}$$

For integer values of M and N, choose $s = 2k + 1$

Examples:

r	***k***	***M***	***N***	$T_{3,M} - T_{6,N}$
$2k$	k	$18k^2 + 2k$	$7k^2 - k$	$[4k(2k+1)]^2$
$3k$	k	$38k^2 + 2k$	$17k^2 - k$	$[6k(2k+1)]^2$
$k+1$	k	$6k^2 + 10k + 4$	$k^2 + 3k + 2$	$[4k^2 + 6k + 2]^2$

Table 1

Case (2):
The choice,

$$2M + 1 = 8r^2 + s^2, \quad 6N - 1 = 8r^2 - s^2 \qquad (4)$$

in (1) leads to the relation that

“$T_{3,M} - 3T_{5,N} =$ **a square integer”**

From (4), the values of ranks of the Triangular numbers and Pentagonal numbers are respectively given by

$$M = \frac{8r^2 + s^2 - 1}{2}, \quad N = \frac{8r^2 - s^2 + 1}{6}$$

For integer values of M and N, choose r and s as follows.

Examples:

r	***s***	***M***	***N***	$T_{3,M} - 3T_{5,N}$
$6k - 4$	$6k - 3$	$162k^2 - 210k + 68$	$42k^2 - 58k + 20$	$[72k^2 - 84k + 24]^2$
$6k$	$6k + 1$	$162k^2 + 6k$	$42k^2 - 2k$	$[12k(6k+1)]^2$

Table 2

Case (3):
The choice,

$$10M - 3 = 8r^2 - s^2, \quad 3(2N + 1) = 8r^2 + s^2 \qquad (5)$$

in (1) leads to the relation that

“$9T_{3,N} - 5T_{7,M} =$ **a square integer”**

From (5), the values of ranks of the Triangular numbers and Heptagonal numbers are respectively given by

$\boxed{M = \frac{8r^2 - s^2 + 3}{10}}$, $\boxed{N = \frac{8r^2 + s^2 - 3}{6}}$

For integer values of M and N, choose r and s as follows.

Examples:

r	s	M	N	$9T_{3,N} - 5T_{7,M}$
$10k-9$	$10k-9$	$70k^2 - 126k + 57$	$150k^2 - 270k + 12$[illegible]	$[200k^2 - 360k + 162]^2$
$10k-1$	$10k-1$	$70k^2 - 14k + 1$	$150k^2 - 30k + 1$	$[200k^2 - 40k + 2]^2$

Table 3

Case (4):
The choice,

$\boxed{2M + 1 = 8r^2 - s^2}$, $\boxed{2N + 1 = 8r^2 + s^2}$ (6)

in (1) leads to the relation that

“$\boxed{\left(Ct_{3,N} - 1\right) - 3T_{3,M} = 3\alpha^2}$”

From (6), the values of ranks of the Centered triangular numbers and Triangular numbers are respectively given by

$\boxed{M = \frac{8r^2 - s^2 - 1}{2}}$, $\boxed{N = \frac{8r^2 + s^2 - 1}{2}}$

For integer values of M and N, choose $\boxed{s = 2k + 1}$

Examples:

r	k	M	N	$\left(Ct_{3,N} - 1\right) - 3T_{3,M}$
$2k$	k	$14k^2 - 2k - 1$	$18k^2 + 2k$	$3[4k(2k+1)]^2$
$3k$	k	$34k^2 - 2k - 1$	$38k^2 + 2k$	$3[6k(2k+1)]^2$
$k+1$	k	$2k^2 + 6k + 3$	$6k^2 + 10k + 4$	$3[4k^2 + 6k + 2]^2$

Table 4

Case (5):
The choice,
$\boxed{4M - 1 = 8r^2 - s^2}$, $\boxed{2N + 1 = 8r^2 + s^2}$ (7)

in (1) leads to the relation that

“$\boxed{\left(Ct_{4,N} - 1\right) - 4T_{6,M} = 4\alpha^2}$”

From (7), the values of ranks of the Centered quadrilateral numbers and Hexagonal numbers are respectively given by

$$\boxed{M = \frac{8r^2 - s^2 + 1}{4}}, \boxed{N = \frac{8r^2 + s^2 - 1}{2}}$$

For integer values of M and N, choose $\boxed{s = 2k + 1}$

Examples:

r	k	M	N	$(Ct_{4,N} - 1) - 4T_{6,M}$
$2k$	k	$7k^2 - k$	$18k^2 + 2k$	$4[4k(2k+1)]^2$
$3k$	k	$17k^2 - k$	$38k^2 + 2k$	$4[6k(2k+1)]^2$
$k+1$	k	$k^2 + 3k + 2$	$6k^2 + 10k + 4$	$4[8k^2 + 12k + 4]^2$

Table 5

4. Conclusion

To conclude, we may search for other relations to (1) by using special polygonal numbers.

5. References

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